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Dynamic Observation Models

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It is shown that radar and quantum mechanics may be modeled using the Kalman–Bucy state-equation observation approach. A method is given for realizing the optimal position filters.

KEY WORDS: Radar; quantum mechanics; nonlinear filter; Doppler shift; optimal filter.

1. INTRODUCTION

In this paper nonlinear state and observation equations are derived for radar and quantum mechanics, and the similarities between the observation mechanisms of these two processes are discussed. A method is then given for obtaining the optimal estimate of the position of a tracked vehicle in the case of radar, or the optimal estimate of the position of a particle in the case of quantum mechanics, conditioned on the past history of the vehicle or particle as seen through their observation equations.

The model for radar assumes a known stochastic nonlinear state equation for the vehicle being tracked. Optimal estimates of the position and velocity of the tracked vehicle are then given using the phase shift, time shift, and Doppler shift of the received radar signal.

While only the observation mechanism of radar is specifically analyzed, the model given here applies to any type of electromagnetic radiation bouncing off a moving body, provided the time shift, phase shift, and Doppler shift of the received signal can be determined.

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An Ito state equation for the tracked vehicle of the form

$$\ddot{x}(t) = f(t, \dot{x}(t), x(t)) + \lambda\dot{z}(t) \tag{1}$$

is assumed given, where the state vector $x(t) = x_1(t), x_2(t), x_2(t)$ is the position of the vehicle, f describes its a priori dynamics, λ is a constant determining roughly the deviation of the vehicle from its deterministic path, and $\dot{z}(t)$ is a three-dimensional white noise process (or the time derivative of a three-dimensional Brownian motion process) with independent coordinates.

It is assumed that f is such that the Ito process (1) is well defined.

It will be shown in the next section that the output of the radar receiver(s) may be modeled as

$$S_i(t + Q_i/c) = h(t - R/c)\sin\{[\omega - (\dot{Q} + \dot{R})/c](t - R/c)\} + \lambda\dot{w}_i \tag{2}$$

where S_i is the output of the ith radar receiver (the ith signal), $h(t)\sin\omega t$ is the signal transmitted by the radar transmitter, and the $\dot{w}_i$ are white noises which are independent of $\dot{z}$. Here R is the range between the vehicle and the radar transmitter, and Q_i is the distance between the target and the ith radar receiver. More than one receiver is allowed so that the phase difference between the received signals and the independence of their noises can be used as additional information for locating the vehicle. Radars of this type are presently being used.

Equations (1) and (2) are the state and observation equations for the radar model presented here. Equation (2) can also model a passive observer as shown in the next section.

Section 3 will show that a model similar to (1) and (2) can be used for quantum mechanical observation. Nelson (Ref. 1, Chapter 13) shows that the path of a particle moving according to quantum mechanical assumptions can be modeled as

$$\dot{x}(t) = b(t, x(t)) + (\hbar/m)^{1/2}\dot{z}(t) \tag{3}$$

where $x(t)$ is the position of the particle at time t, m is its mass, $\hbar$ is Planck's constant divided by 2π, $\dot{z}(t)$ is white noise or the derivative of Brownian motion, and $b(t, x(t))$ is a *real* deterministic function which can, as will be shown in Section 3, be derived from the solution of the appropriate Schrödinger equation. Thus (3) is a state equation analogous to (1). If it is assumed that the quantum mechanical particle is being observed by some kind of electromagnetic radiation, or the particle emits electromagnetic radiation, and that the particle behaves in the usual way toward such radiation, then (3) and (2) are the state and observation equations for the quantum mechanical system.

For the models proposed in Sections 2 and 3, Section 4 shows how to obtain the "optimal" estimates of the state x and the velocity $\dot{x}$ (in the case of radar) at time T given the observations S_i in the interval $[0, T]$, where 0 is the initial time.

Section 5 describes a way of approximating the optimal estimates arbitrarily closely which can be realized in real time in either digital or analog form using the present and past of the observed signals S_i.

Kalman[2] points out that an optimal estimate of $x(T)$ [or $\dot{x}(T)$], conditioned on the observation of the signals to time T, is the conditional mean of $x(T)$ [$\dot{x}(T)$] on the past of the signals. So, the conditional mean will be used here as the estimator of x and $\dot{x}$.

To practically realize this type of filter, the conditional means of x and $\dot{x}$ are first expressed as the ratio of Wiener integrals in Section 4. A real-time method of evaluating this type of Wiener integral is given in Section 5.

Section 6 shows a simplification occurs if the f of (1) is derived via a minimum principle. It is shown that the problem of finding the f that minimizes a certain cost functional may be jointly solved with the problem of finding the best estimate of $x(T)$, given the $S_i(t)$ for $0 \leqslant t \leqslant T$.

The procedure given here for optimally locating a radar-tracked vehicle or a quantum mechanical particle appears to have a number of advantages over present methods.

Presently high-performance radar receivers employ a type of filter called the matched filter. A matched-filter radar receiver has a bank of linear correlators of the form

$$F(d, v) = \int_S^T f(d, v, t) r(t)\, dt$$

where $r(t)$ is the received radar signal and f is the impulse response of a linear filter chosen to maximize the signal-to-noise ratio of the radar receiver. It is easy to show that $f(d, v, t)$ must be chosen so that its Fourier transform is of the form

$$S_0^*(\omega - \Omega) \exp[-i\omega(\tau - t_0)]$$

where $S(\cdot)$ is the Fourier transform (spectrum) of the original transmitted radar pulse, ω is the carrier frequency of the radar, Ω is the Doppler shift due to the relative motion between the radar and the tracked vehicle, τ is time, and t_0 is the delay time for the pulse to go from the radar transmitter to the tracked vehicle and back to the radar receiver. See, for example, Vakman (Ref. 3, p. 5).

A different $f(d, v, t)$ is required for each possible relative velocity between the radar and the tracked vehicle, and sometimes also for each relative distance between the radar and the tracked vehicle. See Vakman[3] and DiFranco and Rubin.[4]

The range and velocity of the tracked vehicle are taken to be the ones that maximize $F(r, v)$. Vakman shows that this procedure is optimal over all filters which choose the maximum of linear correlators. DiFranco and Rubin show that it is optimal from other points of view as well.

The filter described in this paper is optimal (in the manner described in Kalman) over all (linear or nonlinear) functionals T of the form

$$d(t), v(t) = T(t, S_1^*(\cdot), \ldots, S_n^*(\cdot))$$

where $d(t)$, $v(t)$ are the estimated distance and velocity at time t, and the notation $T(t, S_1^*(\cdot), \ldots, S_n^*(\cdot))$ means that the optimal filter takes into account the output of the signals $S_1, \ldots, S_n$ of the n radar receivers from the remote past up to the present time t.

The matched-filter radar gives only one reading of range and range rate per transmitted pulse. The filter described here gives continuous readings, with respect to time, of the estimated position and velocity. The matched-filter approach requires a bank of filters and can give only discrete (with respect now to space) readings of range and range rate. The filter described here, while probably more complicated than a single linear filter, does not require an entire bank of filters. Further, it gives continuous readings with respect to both time and space if an analog realization is used.

If the output of a matched-filter radar is to be Kalman-filtered, then the Kalman filter must be added on after the matched filters. For an example of practical realization of this type of filter see Ref. 5. The Kalman filtering is done in the model presented here as an integral part of the signal filtering.

While both types of filters are optimal, they are optimal over different classes of functions as previously described. The filter presented here is optimal over a wider class of functions than the matched filter and so must be at least as good as the matched filter. It has not as yet been determined if it is better.

The situation in quantum mechanics is similar. In the present treatment of quantum mechanics one is essentially allowed only one reading of position and velocity of the observed particle per experiment (see Ref. 6, Section 5.1). With the filter described here a continuum of readings would be possible if the realization were analog).

As with the radar case, the two approaches might be of the same theoretical accuracy. The filter described here must be at least as accurate as the present observation procedure of quantum mechanics.

Moura *et al.*[7] have also introduced a model similar to the one given for the case of a radar-type passive observer with multiple observation devices. They linearize their equations, however, and use Monte Carlo methods and not analytical methods as done here.

2. THE MODEL FOR RADAR

It is assumed in Section 1 that the radar target has coordinates $x_1(t)$, $x_2(t)$, $x_3(t)$ and has dynamics specified by the stochastic equation (1). For simplicity, it is assumed that the radar transmitter antenna is of the single-beam, nonscanning type. More sophisticated types of transmitting antennas are considered in Ref. 4 and the present analysis can be adapted to them. The radar transmitter is located at u_1, u_2, u_3, and the (possibly) different radar receivers are located at $v_1{}^i$, $v_2{}^i$, $v_3{}^i$. See Fig. 1.

The distance between the target and the radar transmitter at time t is called $R(t)$ and is $\{[x_1(t) - u_1]^2 + [x_2(t) - u_2]^2 + [x_3(t) - u_3]^2\}^{1/2}$.

The distance between the target and the ith receiver at time t is called $Q_i(t)$ and is $\{[v_1{}^i(t) - x_1(t)]^2 + [v_2{}^i(t) - x_2(t)]^2 + [v_3{}^i(t) - x_3(t)]^2\}^{1/2}$.

The local time t at the target will be used as reference time because this approach seems easiest. c is the propagation velocity. If t^* is the time it takes for the transmitted pulse to go from the transmitter to the target, then it follows that $R(t) = ct^*$ and therefore that $t^* = R(t)/c$.

The signal that bounces off the target at time t is, therefore, just before reaching the target, the signal that was transmitted at time $t - t^*$.

It is assumed that the transmitted radar pulse at time $t - t^*$ is of the

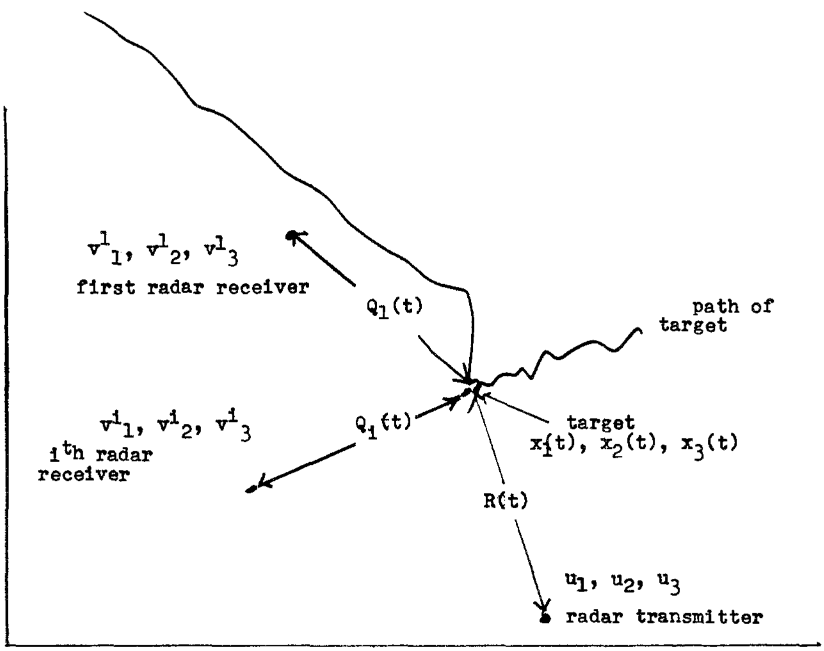

Fig. 1

form $h(t - t^*) \sin \omega(t - t^*)$, where $\sin \omega t$ is the carrier of the radar and $h(\cdot)$ is its modulation.

Thus the signal that reaches the target at time t is

$$h(t - t^*) \sin \omega(t - t^*) = h[t - R(t)/c] \sin \omega[t - R(t)/c]$$

and the signal that bounces off the target is

$$h[t - R(t)/c] \sin\{[\omega - \dot{R}(t)/c][t - R(t)/c]\}$$

where the Doppler effect caused a shift in the carrier frequency of amount $-\dot{R}(t)/c$. We suppose $h(\cdot)$ to be such that the Doppler effect does not significantly change it.

The signal reaches the ith receiver at time $t + Q_i(t)/c$ later, $Q_i(t)/c$ being the time it takes for the signal to travel from the target to the ith receiver. Due to the relative motion between the target and the receiver, the signal is frequency-shifted again. Thus if S_i is the signal at the ith radar receiver, S_i at time $t + Q_i(t)/c$ is

$$S_i\left(t + \frac{Q_i(t)}{c}\right) = h\left(t - \frac{R(t)}{c}\right) \sin\left[\left(\omega - \frac{\dot{R}(t) + \dot{Q}_i(t)}{c}\right)(t - R(t)/c)\right] + \lambda \dot{w}_i(t) \tag{4}$$

where, as in Section 1, $\dot{w}_i$ is white noise. This is the desired equation for the signal at the ith radar receiver.

Section 4 shows how to optimally estimate x_1, x_2, x_3, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ given the signals S_i.

The model here can be changed to cover the case of a passive system as in Ref. 7, by setting $h(t) \equiv 1$ and by assuming that the target generates a monochromatic signal of known frequency.

3. THE MODEL FOR QUANTUM MECHANICS

Nelson (Ref. 1, Chapter 15) shows that the dynamics of a particle of mass m, moving in a potential field V under quantum mechanical assumptions and in one dimension (he also considers the three-dimensional case), can be represented as the solution to a real Ito equation of the form

$$\dot{x}(t) = b(t, x(t)) + (\hbar/m)^{1/2}\dot{z}(t) \tag{5}$$

$b(t, x)$ is determined as follows. The Schrödinger equation for this particular case is

$$(\hbar/t)\psi_t = -(\hbar^2/2m)\psi_{xx} + V\psi \tag{6}$$

Nelson shows that[2]

$$\psi = \exp[R + Si] \tag{7}$$

where

$$R_x = m(b - b_*)/2\hbar \quad \text{and} \quad S_x = m(b + b_*)/2\hbar \tag{8}$$

b is a forward velocity as in (5) and b_* is the backward velocity corresponding to b. Since this theory is quite involved, it is not given here. The reader is referred to Ref. 1 for a good exposition of it and the meaning of b, b_*, forward velocity, and backward velocity.

It is not difficult, however, to derive b from the solution of the Schrödinger equation. On physical grounds and on mathematical grounds[(8–10)] we know for a given particle and potential V that there is a solution ψ to the Schrödinger equation (6). By taking the logarithm of ψ, we can determine the R and S of (7). Differentiating R and S with respect to x, we have from (8) two equations for the unknowns b and b_* which can be solved for b.

The state equation (5) for quantum mechanics is different from the state equation (1) for radar in that (5) is of first order while (1) is of second order. Equations (1) and (5) in fact describe different physical situations. Since $x(t)$ in both processes represents position, $\dot{x}$ in both cases represents velocity. $\dot{x}$ in (5) for fixed x is the sum of a deterministic component, $b(t, x)$, and a white noise component $(\hbar/m)^{1/2}\dot{z}$. Since $\dot{z}$ is violently changing, so is $\dot{x}$. The violent changes of $\dot{x}$ can be thought of as being caused by collisions of the particle (as in Brownian motion) with other particles. These collisions cause jump changes in the velocity $\dot{x}$ while still having the position $x(t)$ continuous in time.

It is not stated here that quantum mechanical effects are caused by collisions of the particle with other particles, only that the distribution of the paths of a quantum mechanical particle has the same distribution as the solutions of (5). Therefore the Ito equation (5) *represents* the distribution of the position of a quantum mechanical particle.

The model (1) has the jump changes in $\ddot{x}$, i.e., in the acceleration or in the force acting on x. Model (5) could be also used for radar if the tracked vehicle were experiencing large, violent changes in velocity.

The mathematics for analyzing the two models is about the same. It is possible to estimate both x and $\dot{x}$ for (1), but only x for (5) since $\dot{x}$ is bouncing back and forth between plus and minus infinity according to the assumption that $\dot{z}$ is white noise. The expected value of $\dot{x}$ given x in (5) is $b(t, x)$ since $E\{\dot{z}\} = 0$. See Doob[(11)] for more on white noise.

The measurement process (4) [or (2)] is not generally used to model

[2] The S and V of this section have no relation to the S and v of the other sections.

quantum mechanical observation. However, there are bodies, signal generators (transmitters) and receivers, such that one is interested in analyzing the quantum mechanical effects on the motion of the bodies, such that the bodies can be illuminated by electromagnetic radiation from the signal generator and there are receivers that can measure the phase, time, and Doppler shift as required by (4). There are also bodies that generate their own electromagnetic radiation, which can be analyzed by a passive phase shift, time shift, and Doppler shift of the type (4).

Thus, while not all quantum mechanical observation models can be put in the form (5), (4), at least some can be.

The potential V of the Schrödinger equation (6) can be thought of as the effect of the measurement apparatus on the motion of the body. Or V can be thought of as consisting of two terms, one of which models the measurement effect and the other of which does not (that is, it comes from the state equation itself).

The noise term $\lambda\dot{w}$ of (4) for radar models the receiver noise. The "noise" in quantum mechanical measurement is generally thought to come from more fundamental processes than the noise in the measurement apparatus. For quantum mechanics, therefore, the $\lambda\dot{w}_i(t)$ term of (4) will consist not only of measurement apparatus noise, but also of noise from more fundamental causes such as the effect of the nonzero wavelength of light.

Vakman[(3)] analyzes quantum mechanics and radar side by side and gives a more complete analogy between the noise in quantum mechanics and the noise in radar.

Noise of the form $\lambda\dot{w}$ does seem to be of a good form for quantum mechanics. It consists of independent spikes of zero width (in time) and of infinite height. It seems that any process which is less erratic or more correlated should be accounted for in the model and therefore not be called noise.

The noise term $(\hbar/m)^{1/2}\dot{z}$ of the state equation (5) is the term that accounts for the inherent randomness of quantum mechanics. This term is not measurement-related and is there irrespective of whether the measurements are taken or not.

4. REPRESENTATION OF THE NONLINEAR FILTERS

A representation of the optimal nonlinear estimate of x (or $\dot{x}$) in the present given S_i in the present and past will now be derived using Wiener integrals for both quantum mechanics and radar.

Let $y(t) = y_1(t), y_2(t), y_3(t)$ be defined by

$$y_1(t) = \dot{x}_1(t), \qquad y_2(t) = \dot{x}_2(t), \qquad y_3(t) = \dot{x}_3(t) \tag{9}$$

for the $x(t)$ of (1). It follows that

$$x_1(T) = \int_0^T y_1(t)\,dt + x_1(0), \qquad x_2(T) = \int_0^T y_2(t)\,dt + x_2(0)$$

$$x_3(T) = \int_0^T y_3(t)\,dt + x_3(0)$$

and

$$\ddot{x}_1(t) = \dot{y}_1(t), \qquad \ddot{x}_2(t) = \dot{y}_2(t), \qquad \ddot{x}_3(t) = \dot{y}_3(t)$$

Equation (1) goes over into

$$\dot{y}(T) = f\left(T, y(T), \int_0^T y(t)\,dt + x(0)\right) + \lambda\dot{z}(T) \tag{10}$$

using vector notation. This Ito equation is of the same general form as (5) except that it has the extra variable $\int_0^T y(t)\,dt + x(0)$. See Ref. 12, p. 479 and Ref. 13 for more on this type of transformation. It follows that the optimal filters for radar and quantum mechanics can be developed with the same mathematics. Let

$$g_i(t, \dot{x}(t), x(t)) = h\left(t - \frac{R(t)}{c}\right) \sin\left[\left(\omega - \frac{\dot{R}(t) + \dot{Q}_i(t)}{c}\right)(t - R(t)/c)\right] \tag{11}$$

From (4) it follows that the observation equation may be written as

$$S_i(t + Q_i(t)/c) = g_i(t, \dot{x}(t), x(t)) + \lambda\dot{w}_i \tag{12}$$

It may be shown (Ref. 12, p. 498) that the conditional mean of $x(T)$ given $S_i(t + Q_i(t)/c)$ with a priori initial density $p(\dot{x}(0), x(0))$ for $\dot{x}$ and $\dot{x}$ may be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y^{\lambda w}\Bigg\{\left[\int_0^T y(t)\,dt + x(0)\right] \exp\Bigg\{-(1/2\lambda^2)\int_0^T \left(f^2 + \sum_i g_i^2\right) dt$$
$$+ (1/\lambda^2)\left[\int_0^T f\,dy(t) + \int_0^T \sum_i g_i S_i(t + Q_i(t)/c)\,dt\right.$$
$$\left.\left. - \tfrac{1}{2}\int_0^T \sum S_i^2(t + Q_i(t)/c)\,dt\right]\Bigg\}\,\Bigg|\, y(0) = \dot{x}_0\Bigg\} p(\dot{x}_0, x_0)\,d\dot{x}_0\,dx_0 \tag{13}$$

divided by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y^{\lambda w}\Bigg\{\exp\Bigg\{-(1/2\lambda^2)\int_0^T \left(f^2 + \sum_i g_i^2\right) dt$$
$$+ (1/\lambda^2)\left[\int_0^T f\,dy(t) + \int_0^T \sum_i g_i S(t + Q_i(t)/c)\,dt\right.$$
$$\left.\left. - \int_0^T \sum_i \tfrac{1}{2} S_i^2(t + Q_i(t)/c)\right]\Bigg\}\,\Bigg|\, y(0) = \dot{x}_0\Bigg\} p(\dot{x}_0, x_0)\,d\dot{x}_0\,dx_0$$

In the above the arguments of f are t, $y(t)$, and $\int_0^T y(t)\,dt + x(0)$ as in (10). The arguments of g are first expressed in terms of R, Q, $\dot{R}$, and $\dot{Q}$ as in (11). These then are expressed in terms of the range and range rate equations of Section 2. Then the x's in these expressions are transformed into y's via Eqs. (9).

The notation $E_y^{\lambda w}\{\cdots \mid y(0) = \dot{x}_0\}$ means expectation with respect to Wiener measure over the paths y with variance λ^2 with the initial value of the paths over which the integration is taken conditioned to be $\dot{x}_0$.

For an exposition of Wiener measure see Kac.[14]

Proof that expression (13) does in fact represent the conditional mean of x, given S, is shown in various forms and under various hypotheses by Schilder (Ref. 12, p. 498), Mortensen,[15] Bucy and Josephs,[16] Zakai[17] and Girsanov.[18]

Equation (13) as written gives the conditional mean of $x(T)$. The conditional mean (optimal estimate) of $\dot{x}(T)$ can be obtained by substituting $y(T)$ for $\int_0^T y(t)\,dt + x_0$ in (13) because of Eqs. (9).

Equation (13) is for radar. For quantum mechanics (5) is used as the state equation and the conditional mean of the state (position of the particle) can be given without using transformations (9) as

$$\begin{aligned} E_x^{\lambda w}\Big\{x(T) \exp\Big\{ &-(1/\lambda^2)\int_0^T \Big(b^2 + \sum_i g_i^2\Big)\,dt \\ &+ (1/\lambda^2)\Big[\int_0^T b\,dx(t) + \int_0^T \sum g_i S(t + Q_i(t)/c) \\ &- \tfrac{1}{2}\int_0^T \sum S_i^2(t + Q_i(t)/c)\,dt\Big]\Big\}\Big|\,x(0) = x_0\Big\} p(x_0)\,dx_0 \end{aligned} \tag{14}$$

divided by

$$\begin{aligned} E_x^{\lambda w}\Big\{\exp\Big\{ &-(1/2\lambda^2)\int_0^T \Big(b^2 + \sum g_i^2\Big)\,dt + (1/\lambda^2)\Big[\int_0^T b\,dx(t) \\ &+ \sum_i g_i S(t + Q_i(t)/c)\,dt \\ &- \tfrac{1}{2}\int_0^T \sum S_i^2(t + Q_i(t)/c)\,dt\Big]\Big\}\Big|\,x(0) = x_0\Big\} p(x_0)\,dx_0 \end{aligned}$$

where for notational convenience λ has been substituted for $(\hbar/m)^{1/2}$ and where the initial distribution $p(x_0)$ is determined in the usual quantum mechanical way.

While the Wiener integrals of (14) are path integrals and do represent a quantum mechanical quantity (the conditional mean), they are not the usual Feynman path integrals. Among other things, they are real and they represent the conditional mean. The Feynman path integrals are usually used to represent the quantum mechanical density.

The conditional density function of a quantum mechanical particle, i.e., the function that gives the probability of finding the particle at any particular place under the assumptions of this paper, can be expressed as the ratio of two real path integrals using the formulas of Mortensen, Bucy, or Zakai as previously listed. If no observations are taken (g as defined above is zero), then their expressions for the density should be the same as would be obtained by the Feynman path integral method.

Actually, models (1) and (4) for radar and (5) and (4) for quantum mechanics are not exactly the models for which representations (13) and (14) have been shown to be true. This is due to the "future" term $Q(t)/c$ in $S(t + Q(t)/c)$ of (4). While this term pushes S into the future, it is actually only a present term since $Q(t)/c$ involves x only in the present. Thus the proofs of (13) and (14) are easily modified to fit this case.

Because of this term, S is actually observed after the motions of the state equations (1) or (5). This of course fits the physical situation.

The $Q(t)/c$ term could be taken out of the S of (4), if there were only one receiver, by a change of time variables of the form $t^{**} = t + Q(t)/c$, which would put the observations in the present and the dynamics [(1) and (5)] in the past. Representations (13) and (14) could then be rigorously derived for t^{**} time. The noises $\dot{\omega}$ and $\dot{z}$ would not be affected by this time transformation since noise by definition (see McKean[19]) is translation invariant.

However, this approach would involve time translations through Ito differentials, which involves a great risk of mistake (since nonlinear equations are involved) and probably subjective interpretations on the order of the various limits.

Thus the approach of this paper is to keep vehicle, particle time as present and derive all results with respect to this reference.

As will be shown in the next section and as seen from expressions (13) and (14), it is possible to derive optimal estimates for the conditional mean of x or $\dot{x}$ in the present given S in the future, present, and past. Therefore no theoretical difficulties are involved if it is assumed that a reading of S is taken over all time and then if one goes back and computes the optimal value of x at a particular time T.

No practical difficulties are involved either with this approach due to the large size of c in relation to the other constants. While S may be required into the future, it is not required very far into the future. (See Section 5.) Once a practical filter has been devised, one takes observed time as present and the dynamics time as past.

While unfortunately expressions (13) and (14), which express the best estimate of the state x given the observed signals, are quite complicated, it should be remembered that they are the exact best estimates of the state, given the observations. The only approximation made so far is that the Doppler effect only affects the carrier of the transmitted signal $h(\cdot)\sin\omega(\cdot)$. It is also shown that quantum mechanical observation can be modeled with entirely classical principles.

5. REALIZATION OF THE OPTIMAL FILTERS

This section describes a numerical method for approximating the path integral expressions (13) and (14). Other numerical methods for realizing nonlinear filters are given in Ref. 20.

The present method is a variant of the perturbation expansion of quantum mechanics which is explained in detail in Ref. 6, Chapter 6. This method was used to realize an optimal FM filter in Ref. 21.

The perturbation expansion has a number of desirable features. As will be shown, it can be realized in real time. This is particularly important for radar. It can be realized in either digital or analog form, a factor which might be of value in reducing the size, weight, and cost of some radar sets. The absolute error committed by any approximate realization can be bounded. Each term of the expansion to be given has physical significance in either the radar or quantum mechanical applications (see Ref. 6, Chapter 6).

The filter is also most accurate in low-signal-to-noise ratio cases—the area where best performance is desired.

The expansion will be carried out only on the numerator of (13) since the denominator of (13) is the same as the numerator with the $\int_0^T y(t)\,dt + x_0$ term set equal to one. Equation (14) is a special case of (13) since b's of (14) have one less variable than the f's of (13), as pointed out previously.

For simplicity, the expansion will be carried out for vehicles moving in only one dimension and for one receiver. Expansion in higher dimensions requires no concepts that are not used in the one-dimensional case and the extra notation tends to obscure the meaning of the operations.

A further specialization of the processes considered is that their initial position and velocity are normally distributed with means $\bar{x}$, $\bar{\dot{x}}_0$, variances σ_P^2, σ_V^2, and covariance ρ, and also that the initial values are independent of the perturbing Brownian motion.

The reason for this is mostly that if definite assumptions are made about the initial values of the processes, the expansion can be carried out further. Also, the normal case is the easiest to work with and the case usually used in practice. Since the initial density function p of (13) and (14) has been specialized to the normal density described above, the notation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_y^{\lambda w}\{\cdots\,|\,y(0) = \dot{x}_0\}p(x_0, \dot{x}_0)\,dx_0\,d\dot{x}_0$$

meaning expectation with respect to Wiener measure with initial distribution given by p, will be replaced by the simpler and more usual notation

$$E_y^{\lambda w}\{\cdots\}$$

meaning the same as above.

A lemma is now needed.

Lemma 5.1.[3] If $y(t)$ is Brownian motion with variance λ^2 and with $y(0) = \dot{x}_0$, and x_0, $\dot{x}_0$ are normally distributed with means $\bar{x}_0$, $\bar{\dot{x}}_0$ and variances σ_P^2, σ_V^2 and covariance ρ, then $\int_0^t y(\alpha)\,d\alpha + x_0$, $y(t)$, and x_0 are normally distributed with means $t\bar{\dot{x}}_0 + \bar{x}_0$, $\bar{\dot{x}}_0$, and $\bar{x}_0$, and with variances $\lambda^2 t^3/3 + (t\sigma_V)^2 + t\rho + \sigma_P^2$, $t\lambda^2 + \sigma_V^2$, and σ_P^2. The covariance between $\int_0^t y(\alpha)\,d\alpha$ and $y(t)$ is $(\lambda t)^2/2 + t\sigma_V^2 + \rho$. The covariance between $y(t)$ and x_0 is ρ. The covariance between $\int_0^t y(\alpha)\,d\alpha + x_0$ and x_0 is $t\rho + \sigma_P^2$.

Proof. $y(t)$ can be written $y(t) = y(t) - y(0) + y(0) = y(t) - y(0) + \dot{x}_0$. By definition $y(t) - y(0)$ (this is Brownian motion with initial value zero) has mean zero and variance $t\lambda^2$. By assumption $\dot{x}_0$ is normal with mean $\bar{\dot{x}}_0$ and variance σ_V^2. Thus $y(t)$ is normal with mean $\bar{\dot{x}}_0$ and variance $t\lambda^2 + \sigma_V^2$. Now, $\int_0^T y(\alpha)\,d\alpha + x_0$ is normal since it is the limit of sums of normal variables. Since Wiener measure is the measure associated with Brownian motion, it follows that the mean of $\int_0^t y(\alpha)\,d\alpha + x_0$ is

$$\begin{aligned} E_y^{\lambda w}\left\{\int_0^t y(\alpha)\,d\alpha + x_0\right\} &= E_y^{\lambda w}\left\{\int_0^t [y(\alpha) - y(0)]\,d\alpha + t\dot{x}_0 + x_0\right\} \\ &= \int_0^t E_y^{\lambda w}\{[y(\alpha) - y(0)]\}\,d\alpha + t\bar{\dot{x}}_0 + \bar{x}_0 \\ &= t\bar{\dot{x}}_0 + \bar{x}_0 \end{aligned}$$

and the variance of $\int_0^t y(\alpha)\,d\alpha + x_0$ is

$$\begin{aligned} &E_y^{\lambda w}\left\{\left[\int_0^t y(\alpha)\,d\alpha + x_0 - t\bar{\dot{x}}_0 - \bar{x}_0\right]^2\right\} \\ &\quad = E_y^{\lambda w}\left\{\left[\int_0^t (y(\alpha) - y(0))\,d\alpha + t(y(0) - \bar{\dot{x}}_0) + (x_0 - \bar{x}_0)\right]^2\right\} \end{aligned}$$

Since the Brownian motion is independent of the initial distribution, this can be written as

$$\begin{aligned} &E_y^{\lambda w}\left\{\left[\int_0^t (y(\alpha) - \dot{x}_0)\,d\alpha\right]^2\right\} + t^2\sigma_V^2 + t\rho + \sigma_P^2 \\ &\quad = E_y^{\lambda w}\left\{\int_0^t (y(\alpha_1) - \dot{x}_0)\,d\alpha_1 \int_0^t (y(\alpha_2) - \dot{x}_0)\,d\alpha_2\right\} + t^2\sigma_V^2 + t\rho + \sigma_P^2 \\ &\quad = \int_0^t \int_0^t E_y^{\lambda w}\{(y(\alpha_1) - \dot{x}_0)(y(\alpha_2) - \dot{x}_0)\}\,d\alpha_1\,d\alpha_2 + t^2\sigma_V^2 + t\rho + \sigma_P^2 \end{aligned}$$

[3] See Ref. 11, p. 651, for background on Brownian motion.

Since $\min(\alpha_1, \alpha_2)$ (the minimum of α_1 and α_2) is the covariance function of Brownian motion, this last expression is

$$\begin{aligned}
\lambda^2 \int_0^t \int_0^t \min(\alpha_1, \alpha_2)\, d\alpha_1\, d\alpha_2 + \sigma_V{}^2 t^2 + t\rho + \sigma_P{}^2 \\
= \lambda^2 \int_0^t \left[\int_0^{\alpha_2} \min(\alpha_1, \alpha_2)\, d\alpha_1 + \int_{\alpha_2}^t \min(\alpha_1, \alpha_2)\, d\alpha_1 \right] d\alpha_2 \\
+ t^2 \sigma_V{}^2 + t\rho + \sigma_P{}^2 \\
= \lambda^2 \int_0^t \left[\int_0^{\alpha_2} \alpha_1\, d\alpha_1 + \int_{\alpha_2}^t \alpha_2\, d\alpha_1 \right] d\alpha_2 + \sigma_V{}^2 t^2 + \rho t + \sigma_P{}^2 \\
= \lambda^2 \int_0^t [\alpha_2{}^2/2 + \alpha_2(t - \alpha_2)]\, d\alpha_2 + t^2\sigma_V{}^2 + t\rho + \sigma_P{}^2 \\
= \lambda^2 t^3/3 + t^2 \sigma_V{}^2 + t\rho + \sigma_P{}^2
\end{aligned}$$

In the same way the covariance of $\int_0^t y(\alpha)\, d\alpha + x_0$ and $y(t)$ is

$$\begin{aligned}
E_y^{\lambda w}\left\{ \left[\int_0^t y(\alpha)\, d\alpha + x_0 - t\bar{x}_0 - \bar{x}_0 \right] [y(t) - \bar{x}_0] \right\} \\
= E_y^{\lambda w}\left\{ \left[\int_0^t (y(\alpha) - y(0))\, d\alpha + t(y(0) - \bar{x}_0) + (x_0 - \bar{x}_0) \right] \right. \\
\left. \times [(y(t) - y(0)) + (y(0) - \bar{x}_0)] \right\} \\
= E_y^{\lambda w}\left\{ \int_0^t (y(\alpha) - y(0))(y(t) - y(0))\, d\alpha \right\} + t\sigma_V{}^2 + \rho \\
= \lambda^2 \int_0^t \min(\alpha, t)\, d\alpha + t\sigma_V{}^2 + \rho \\
= \lambda^2 \int_0^t \alpha\, d\alpha + t\sigma_V{}^2
\end{aligned}$$

The covariances between x_0 and $\int_0^t y(\alpha)\, d\alpha + x_0$ and $y(t)$ are calculated in the same manner.

Now let I be

$$\begin{aligned}
I = E_y^{\lambda w}\left\{ \left[\int_0^T y(t)\, dt + x_0 \right] \exp\left\{ (-1/2\lambda^2) \int_0^T (f^2 + g^2)\, dt \right. \right. \\
\left. \left. + (1/\lambda^2) \left[\int_0^T f\, dy(t) + \int_0^T gS(t + Q(t)/c)\, dt^{-1/2} \int_0^T S^2(t + Q(t)/c)\, dt \right] \right\} \right\}
\end{aligned}$$

Thus I is the numerator of (13) and the path integral to be evaluated. Note that this path integral has neither end point of the paths fixed, while the path integrals of (6) have both end points of the paths fixed.

The procedure for evaluating I is to expand the exp term in a power series and calculate the series term by term. I becomes

$$I = \sum_n (1/n!) E_y^{\lambda w}\left\{\left(\left[\int_0^T y(t)\,dt + x_0\right]\left((-1/2\lambda^2)\int_0^T (f^2 + g^2)\,dt\right.\right.\right.$$
$$\left.\left.\left. + (1/\lambda^2)\left[\int_0^T f\,dy(t) + \int_0^T gS(t + Q(t)/c)\,dt^{-1/2}\int_0^T S^2(t + Q(t)/c)\,dt\right]\right)^n\right)\right\}$$

The interchange of the sum and integral here and in other places can be rigorously justified as in Kac.[22]

The zeroth term of I is

$$E_y^{\lambda w}\left\{\int_0^T y(t)\,dt + x_0\right\} = T\bar{x}_0 + \bar{x}_0$$

by Lemma 5.1.

We now consider the following part of the first term,

$$J = E_y^{\lambda w}\left\{x_0\left[(-1/2\lambda^2)\int_0^T (f^2 + g^2)\,dt\right.\right.$$
$$\left.\left. + (1/\lambda^2)\left[\int_0^T f\,dy(t) + \int_0^T gS(t + Q(t)/c)\,dt\right]\right]\right\}$$

The expected value of the Ito integral $\int_0^T f\,dy(t)$ is zero, by Dynkin (Ref. 23, p. 211), since dy is independent of x_0.

The functions g and Q of the path integral part of J will now be specialized according to their previous definitions. J will then be expressed as a finite-dimensional integral which can be numerically evaluated. According to (11), g is

$$g(t, \dot{x}(t), x(t)) = h(t - R(t)/c)\sin\{[\omega - (\dot{R} + \dot{Q})/c][t - R(t)/c]\}$$

where R, the distance between the radar transmitter and the target, is, for one-dimensional motion (Section 2),[4] $R(t) = |x(t) - u|$ and where Q, the distance between the target and the radar receiver, in one dimension is $Q = |x(t) - v|$.

Under transformation (9), R and Q go over into

$$R(t) = \left|\int_0^t y(\alpha)\,d\alpha + x_0 - u\right|, \qquad Q(t) = \left|\int_0^t y(\alpha)\,d\alpha + x_0 - v\right| \quad (15)$$

[4] While absolute values do not appear in two or more dimensions, their treatment is still instructive.

Differentiating (15) with respect to t, we get

$$\dot{R}(t) = \pm y(t) \qquad \text{and} \qquad \dot{Q}(t) = \pm y(t) \tag{16}$$

where the plus sign is used for $\dot{R}$ if $\int_0^t y(\alpha)\, d\alpha + x_0 - u > 0$, and the minus sign is used for $\dot{R}$ if $\int_0^t y(\alpha)\, d\alpha + x_0 - u < 0$. The same applies to $\dot{Q}$. Putting these expressions into J, we get

$$\begin{aligned} J = \int_0^T E_y^{\lambda w}\Bigg\{ & x_0 \frac{-1}{2\lambda^2}\Bigg\{\Bigg[f\Bigg(t, y(t), \int_0^t y(\alpha)\, d\alpha + x_0\Bigg)\Bigg]^2 \\ & + \Bigg(h\Bigg(t - \frac{\left|\int_0^t y(\alpha)\, d\alpha + x_0 - u\right|}{c}\Bigg)\Bigg)^2 \\ & \times \Bigg(\sin\Bigg[\Bigg(\omega - \frac{\pm y(t) \pm y(t)}{c}\Bigg)\Bigg(t - \frac{\left|\int_0^t y(\alpha)\, d\alpha + x_0 - u\right|}{c}\Bigg)\Bigg]\Bigg)^2\Bigg\} \\ & + \frac{1}{\lambda^2} h\Bigg(t - \frac{\left|\int_0^t y(\alpha)\, d\alpha + x_0 - u\right|}{c}\Bigg) \\ & \times \sin\Bigg[\Bigg(\omega - \frac{\pm y(t) \pm y(t)}{c}\Bigg)\Bigg(t - \frac{\left|\int_0^t y(\alpha)\, d\alpha + x_0 - u\right|}{c}\Bigg)\Bigg] \\ & \times S\Bigg(t + \frac{\left|\int_0^t y(\alpha)\, d\alpha + x_0 - v\right|}{c}\Bigg)\Bigg\}\, dt \end{aligned}$$

For fixed t now the random variables $\int_0^t y(\alpha)\, d\alpha + x_0$, $y(t)$, and x_0 become ordinary one-dimensional random variables whose distribution is given by Lemma 5.1. Letting $\int_0^t y(\alpha)\, d\alpha + x_0 = a$ and $y(t) = b$, and using the result in Ref. 24 (since the result there is for zero means, the nonzero means must be added in as done here), the function space integral J reduces to the following fourfold integral:

$$\begin{aligned} J = -\int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} & [(2\pi)^3 \det M]^{-1/2} \frac{x_0}{\lambda^2} \\ \times \Bigg[\Bigg\{ & \frac{[f(t, b, a)]^2}{2} + \frac{1}{2}\Bigg\{h\Bigg(t - \frac{|a - u|}{c}\Bigg) \\ & \times \sin\Bigg[\Bigg(\omega - \frac{\pm b \pm b}{c}\Bigg)\Bigg(t - \frac{|a - u|}{c}\Bigg)\Bigg]\Bigg\}^2\Bigg\} \\ & - \frac{1}{\lambda^2} h\Bigg(t - \frac{|a - u|}{c}\Bigg) \sin\Bigg[\Bigg(\omega - \frac{\pm b \pm b}{c}\Bigg)\Bigg(t - \frac{|a - u|}{c}\Bigg)\Bigg] S\Bigg(t + \frac{|a - v|}{c}\Bigg)\Bigg] \\ & \times \exp\{-\tfrac{1}{2}[(a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0)] \\ & \times M^{-1}[((a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0))]\}\, da\, db\, dx_0\, dt \end{aligned}$$

M is the covariance matrix calculated in Lemma 5.1:

$$M = \begin{bmatrix} \lambda^2 t^3/3 + (t\sigma_V)^2 + t\rho + \sigma_P{}^2 & (\lambda t)^2/2 + t\sigma_V{}^2 + \rho & t\rho + \sigma_P{}^2 \\ (\lambda t)^2/2 + t\sigma_V{}^2 + \rho & t\lambda^2 + \sigma_V{}^2 & \rho \\ t\rho + \sigma_P{}^2 & \rho & \sigma_P{}^2 \end{bmatrix}$$

There are four possibilities for the terms with the absolute value signs [these four possibilities also determine the signs in (16)]:

I. $a - u > 0$ and $a - v > 0$
II. $a - u < 0$ and $a - v > 0$
III. $a - u > 0$ and $a - v < 0$
IV. $a - u < 0$ and $a - v < 0$

which go over into

I. $a > u$ and $a > v$
II. $a < u$ and $a > v$
III. $a > u$ and $a < v$
IV. $a < u$ and $a < v$

Thus if $v < u$ (u and v being the one-dimensional positions of the radar transmitter and receiver; see Fig. 1) we have three regions of a integration,

$$a > u, \qquad \text{or} \qquad v < a < u, \qquad \text{or} \qquad a < v$$

The a integration (where a is the position of the target) is done separately on each of these regions.

The $\pm$ signs in the last expression can be removed and also the absolute value signs. J is then ready for evaluation or numerical integration.

Whether or not the terms of I not containing the received signal S can be integrated in closed form [f^2 and $(h \sin)^2$ in this case] depends on the form of f and h. If f and h are of a fairly simple form, such as polynomials, some trigonometric functions, exponentials, or step functions, direct evaluation is possible, as shown in Lemma 5.2, due to the Gaussian density function. If not, then some sort of numerical approximation can be used, as will be shown.

Due to the unknown nature of the received signal S, terms containing S cannot be completely integrated in closed form. It can be seen that the series in the signal terms is exactly the same as the series considered by Wiener,[25] McKean,[26] and King[27] except that here S is the received signal and not white noise.

A lemma is now given which shows how to evaluate the linear filters appearing in J.

Lemma 5.2.

$$[(2\pi)^3 \det M]^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_0 \sin(bE + F)$$
$$\times \exp[-\tfrac{1}{2}((a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0))$$
$$\times M^{-1}((a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0))]\, db\, dx_0$$
$$= (2\pi M_{11})^{-1/2}\{[M_{13}(a - t\bar{x}_0 - \bar{x}_0)/M_{11} + \bar{x}_0]$$
$$\times \sin[(a - t\bar{x}_0 - \bar{x}_0)M_{12}E/M_{11} + F + \bar{x}_0 E]$$
$$+ [(M_{11}M_{32} - M_{12}M_{31})E/M_{11}]$$
$$\times \cos[(a - t\bar{x}_0 - \bar{x}_0)M_{12}E/M_{11} + F + \bar{x}_0 E]\}$$
$$\times \exp[-(a - t\bar{x}_0 - \bar{x}_0)^2/2M_{11} - E^2(M_{11}M_{22} - M_{12}^2)/2M_{11}]$$

where M_{ij} and M denote the covariance matrix calculated in Lemma 5.1 and E and F are constants.

Proof. The proof is done by brute force. The inverse matrix is given a name and all the factors in the exponent are explicitly written out. The sin term is written as the imaginary part of $\exp[i(bE + F)]$. All terms in the exponent are now of at most second order. The operation of taking the imaginary part is taken outside the integral signs, the square on b in the exponent is completed, and then b is integrated out. The square is then completed on x_0 and x_0 is integrated out. The imaginary part of the result is given in the conclusion of the lemma.

J is now separated into six different parts. First J is separated into the three regions of a integration, $a > u$; $v < a < u$; and $a < v$. The absolute value signs are removed and the $\pm$ signs fixed. Then each of the three integrals is split into two integrals—a signal part (containing S), and a nonsignal part.

Since evaluation of the nonsignal integrals is similar to the evaluation of the signal part integrals and since further evaluation of these integrals requires specialization of the f and h functions, they will not be further considered.

The region $v < a < u$ is considered first for the signal part of J. Thus for this case $|a - u| = -(a - u)$, $|a - v| = a - v$, and the Doppler term $(\pm b \pm b)$ cancels out by the reasoning behind Lemma 5.2 or by the physical reason that for this case the target is between the transmitter and the receiver. Thus this part of the J integral is

$$\lambda^{-2} \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_v^u [(2\pi)^3 \det M]^{-1/2} x_0 h(t + (a - u)/c) \sin\{\omega[t + (a - u)/c]\}$$
$$\times S(t + (a - v)/c) \exp[-\tfrac{1}{2}((a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0))$$
$$\times M^{-1}((a - t\bar{x}_0 - \bar{x}_0), (b - \bar{x}_0), (x_0 - \bar{x}_0))]\, da\, db\, dx_0\, dt$$

which by Lemma 5.2 {with $E = 0$ and $F = \omega[t + (a - u)/c]$} is

$$\lambda^{-2} \int_0^T \int_v^u (2\pi M_{11})^{-1/2}[M_{13}(a - t\bar{x}_0 - \bar{x}_0)/M_{11} + \bar{x}_0]h(t + (a - u)/c) \\ \times \sin\{\omega[t + (a - u)/c]\} S(t + (a - v)/c) \\ \times \exp[-(a - t\bar{x}_0 - \bar{x}_0)^2/2M_{11}]\, da\, dt \tag{17}$$

which is of the general form

$$\int_0^T \int_v^u D(a, t)S(t + (a - v)/c)\, da\, dt$$

This shows that (17) can be considered to be a linear filter on S. Evaluation in real time is possible since the different components of D in multidimensional target spaces can be integrated independently (this would not be the case for multidimensional observation spaces) and also since D decays quickly due to the exponential.

If time is measured in seconds and distance in miles, then c is 186,000 and the maximum of $u - v$ is on the order of 100 and therefore the $(a - v)/c$ type terms in (17) have a maximum of about 1/1860. Thus another way of approximating (17) is to make a Taylor series expansion about $(a - v)/c = 0$.

Since $(a - v)/c$ is so small, one might propose to leave it out entirely. If this criterion is applied equally to all terms of J, however, J would not depend on u and v, the positions of the transmitter and receiver, and this would lead to a meaningless result. The slight dependence of the estimated state on the u and v variables is not a weakness of the present analysis; it is inherent in determining the position of a body by measuring the time, phase, and Doppler shift of the electromagnetic waves it reflects.

Therefore, a Taylor series expansion of (17) must go out to at least the first term. A second seems unnecessary since it would be of order 1/3,000,000.

If this Taylor series is made, then the a integration of (17) can be easily carried out. A difficulty arises, however, because the signal S must also be differentiated and according to (4), S is the sum of a drift component and a white noise component $\lambda\dot{w}$. White noise is the derivative of Brownian motion, which, according to a number of criteria, does not exist (see Ref. 11). Therefore the observation model (4) must be changed to allow for the differentiation of S.

The extra hypothesis is now added that while the signal as it reaches the antenna is described by (4), when it is ready for processing by the present filter the signal has been preprocessed by the radar receiver so that S and its derivative exist. This appears to be a reasonable hypothesis since this type of preprocessing must go on in any practical receiver. This hypothesis is equivalent to the assumption that the receiver does not pass signals of arbitrarily high frequency.

If this hypothesis is made, then all terms of I (defined after Lemma 5.1) can be integrated. The derivative of S always appears as being integrated next to another function, and therefore integration by parts is always possible, so that the filters need depend only on S and not on the derivative of S.

The filter given in this paper has the advantage of being recursive, i.e., as new signals come in they can be processed starting from the present—it is not necessary to reprocess all the old signals.

The term (17) depends on S in the future of T. However, it does not depend on S very far into the future. From previous considerations and (17) it follows that (17) depends on S at most 1/1860 sec into the future.

The term just covered was for $v < a < u$. We now consider the case $a > u$, which corresponds to the transmitter situated between the target and the receiver.

As was done with the first term, let a denote (random) target position and b denote (random) target velocity. Using Lemma 5.1, the signal part of this term of J becomes

$$\begin{aligned}\lambda^{-2}\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_u^{\infty} &[(2\pi)^3 \det M]^{-1/2}x_0 h(t-(a-u)/c)\\ &\times \sin[(\omega - 2b/c)(t-(a-u)/c)]\, S(t+(a-v)/c)\\ &\times \exp\{-\tfrac{1}{2}[(a-t\bar{x}_0-\bar{x}_0), (b-\bar{x}_0), (x_0-\bar{x}_0)]\\ &\times M^{-1}[(a-t\bar{x}_0-\bar{x}_0), (b-\bar{x}_0), (x_0-\bar{x}_0)]\}\, da\, db\, dx_0\, dt\end{aligned}$$

which by Lemma 5.2 {with $E = (-2/c)[t-(a-u)/c]$ and $F = \omega[t-(a-u)/c]$} is

$$\begin{aligned}\lambda^{-2}(2\pi M_{11})^{-1/2}&\int_0^T\int_u^{\infty} h\left(t-\frac{a-u}{c}\right)S\left(t+\frac{a-v}{c}\right)\left\{\left(M_{13}\frac{a-t\bar{x}_0-\bar{x}_0}{M_{11}}+\bar{x}_0\right)\right.\\ &\times \sin\left[-2(a-t\bar{x}_0-\bar{x}_0)M_{12}\left(t-\frac{(a-u)/c}{cM_{11}}\right)\right.\\ &\left.+\,\omega\left(t-\frac{a-u}{c}\right)-2\bar{x}_0\frac{t-(a-u)/c}{c}\right]\\ &+\frac{M_{11}M_{32}-M_{12}M_{31}}{M_{11}}\left(-2\frac{t-(a-u)/c}{c}\right)\\ &\times \cos\left[\frac{a-t\bar{x}_0-\bar{x}_0}{M_{11}}\left(-2M_{12}\frac{t-(a-u)/c}{c}\right)\right.\\ &\left.\left.+\,\omega\left(t-\frac{a-u}{c}\right)-2\bar{x}_0\frac{t-(a-u)/c}{c}\right]\right\}\\ &\times \exp\left[-\frac{(a-t\bar{x}_0-\bar{x}_0)^2}{2M_{11}}-4\left(t-\frac{a-u}{c}\right)^2\frac{M_{11}M_{22}-M_{12}^2}{c^2M_{11}}\right] da\, dt\end{aligned}$$

The last signal term of J $(a < v)$ is, using Lemma 5.2 with $E = (2/c)[t + (a - u)]/c$ and $F = \omega[t + (a - u)/c]$,

$$\lambda^2(2\pi M_{11})^{-1/2}\int_0^T\int_{-\infty}^{v} h\left(t + \frac{a-v}{c}\right)S\left(t - \frac{a-v}{c}\right)\left\{\left[\frac{M_{13}(a - t\bar{x}_0 - \bar{x}_0)}{M_{11}} + \bar{x}_0\right]\right.$$

$$\times \sin\left[2(a - t\bar{x}_0 - \bar{x}_0)M_{12}\frac{t + (a-u)/c}{cM_{11}}\right.$$

$$\left. + \omega\left(t + \frac{a-u}{c}\right) + 2\bar{x}_0\frac{t + (a-u)/c}{c}\right]$$

$$+ \frac{M_{11}M_{32} - M_{12}M_{31}}{M_{11}}\left(2\frac{t + (a-u)/c}{c}\right)$$

$$\times \cos\left[2(a - t\bar{x}_0 - \bar{x}_0)M_{12}\frac{t + (a-u)/c}{c}\right.$$

$$\left.\left. + \omega\left(t + \frac{a-u}{c}\right) + 2\bar{x}_0\frac{t + (a-u)/c}{c}\right]\right\}$$

$$\times \exp\left[\frac{-(a - t\bar{x}_0 - \bar{x}_0)^2}{2M_{11}} - 4\left(t + \frac{a-u}{c}\right)^2\frac{M_{11}M_{22} - M_{12}^2}{c^2M_{11}}\right] da\, dt$$

These expressions for the linear filters on S are not as complicated as they might seem, since they involve a to at most a second-order term in the exponent and a to at most a linear term outside the exponent. Dependence on t (the other variable of integration) is more complicated since the covariance terms M_{ij} depend on it, but the dependence on t is still only in terms of relatively simple rational functions.

The two terms above involve S into the distant future. On the assumption that h and the signal are bounded, however, which they will always be in a practical system, both the above terms can be truncated after a (small) finite time if the inequality on the tails of a normal distribution given by Feller [Ref. 28, Eq. (VII 1.8)] is used.

This completes the analysis of J.

Another part of the first term of I is

$$E_y^{\lambda w}\left\{\left[\int_0^T y(t)\, dt\right]\left[(-1/2\lambda^2)\int_0^T (f^2 + g^2)\, dt\right.\right.$$

$$\left.\left. + (1/\lambda^2)\left[\int_0^T f\, dy(t) + \int_0^T gS(t + Q/c)\, dt\right]\right]\right\}$$

The order of integration is changed to

$$\int_0^T\int_0^T E_y^{\lambda w}\{y(t_1)[(-1/2\lambda^2)(f^2 + g^2) + (1/\lambda^2)gS(t_2 + Q(t_2)/c)]\}\, dt_1\, dt_2$$

$$+ (1/\lambda^2)\int_0^T E_y^{\lambda w}\{y(t_1)\int_0^T f\, dy(t_2)\}\, dt_1$$

The joint distribution (the covariance matrix) of the normal variables $y(t_1)$, $y(t_2)$ and $\int_0^{t_2} y(\alpha)\, d\alpha + x_0$ is now calculated and the above function space integral reduces to an ordinary fourfold integral as J did. The Ito integral does not drop out as in the previous case since evidently $y(t_1)$ and $dy(t_2)$ are not independent. This term is calculated by letting $y(t_1) = \int_0^{t_1} dy(\alpha)$ and using Eq. (7.6) in Dynkin[23] (see also Wiener,[25] first part of lecture 2; and Schilder, Ref. 12, Lemma 2.2).

Calculation of the higher order terms proceeds in a similar manner.

The estimates necessary to prove convergence of the I series in Kac[22] indicate that more and more terms are required as time increases. However, examples[21] indicate the reverse, an exponentially fast decay into a steady state whose analogs in the linear case are the stationary processes first studied by Wiener.[29]

The criteria for determining whether the series given here converges with a few terms independently of time or blows up with increasing time is probably whether the x process (1) itself is stable or not (see Kushner[30] for a definition of stochastic stability). Inclusion of boundary conditions (Schilder[21,31]; Dynkin,[23] p. 115, Vol. II) keeps the process from going to infinity.

The received signal for the present case is $g(t, \dot{x}(t), x(t)) + \lambda\dot{w}$ [see (10)]. This can be written in the more general form as $Ag(t, \dot{x}(t), x(t)) + \lambda\dot{w}$, where A is a measure of the signal power and λ is a measure of the noise power, and therefore A/λ^2 is proportional to the signal-to-noise ratio for the present system. If g is replaced wherever it appears in the series I by Ag, it can be seen that part of the series for the conditional mean is a power series in the signal-to-noise ratio A/λ^2. Thus the filter presented here can be viewed as a power series expansion in the signal-to-noise ratio having the greatest accuracy and requiring the least number of terms in the case when the signal-to-noise ratio is small.

Just as power series in complex variables can be rearranged to have different circles on convergence, so can the series given here in the signal-to-noise ratio. It is possible to rearrange it in such a manner that the fourth term of I (for example) will depend only on the fourth power of the signal-to-noise ratio.

6. A STOCHASTIC HAMILTON'S PRINCIPLE

If it can be assumed that the radar-tracked vehicle is moving in such a manner that a cost functional is minimized, then calculation of the J integral is simplified.

If the f of (1) is determined in such a manner that the expected value [with respect to (1)] of

$$\int_0^T \{L(t, \dot{x}(t), x(t)) + [f(t, \dot{x}(t), x(t))]^2/2\}\, dt + \phi(\dot{x}(T), x(T)) \qquad (18)$$

is minimized, then (13) becomes

$$E_y^{\lambda w}\Biggl\{\Biggl[\int_0^T y(t)\, dt + x(0)\Biggr] \exp\Biggl[-\tfrac{1}{2}\int_0^T (L + \textstyle\sum g_i^2)\, dt \times (1/\lambda^2)\Biggl[\int_0^T \sum g_i S_i(t + Q_i/c)\, dt + \int_0^T \sum S_i^2(t + Q_i(t)/c)\, dt + \phi(\dot{x}(T), x(T))\Biggr]\Biggr]\Biggr\} \qquad (19)$$

divided by

$$E_y^{\lambda w}\Biggl\{\exp\Biggl[(-1/2\lambda^2)\int_0^T (L + \textstyle\sum g_i^2)\, dt + \int_0^T (1/\lambda^2) \sum_i g_i S_i\, dt + \int_0^T \sum_i S_i^2(t + Q(t)/c)\, dt + \phi\Biggr]\Biggr\}$$

Proof of (19) is the same as the proofs of Theorems 3.2 and 3.3 of Ref. 12. See Kushner[31] for other derivations of the optimal control equations.

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